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LIMIT THEOREMS FOR THE SIMPLE BRANCHING PROCESS ALLOWING IMMIGRATION, II: THE CASE OF INFINITE OFFSPRING MEAN

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A B S T R A C T

This paper obtains some limit theorems for the simple branching process allowing immigration $\{X_n\}$ when the offspring mean is infinite. It is shown that there exists a function U such that $\{e^{-n}U(X_n)\}$ converges almost surely and if $s = \Sigma b_j \log^+ U(j) < \infty$, where $\{b_j\}$ is the immigration distribution, the limit is non-defective and non-degenerate but is infinite if $s = \infty$.

When $s = \infty$ limit theorems are found for $\{U(X_n)\}$ which involve a slowly varying non-linear norming.

Key words: Bienaymé-Galton-Watson branching process, immigration, martingale convergence, limit theorems, regular variation.

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1. INTRODUCTION

Let $\{X_n:n=0,1,\ldots\}$ denote a Bienaymé-Galton-Watson process with immigration, for which the offspring probability generating function, f, satisfies $m = f'(1-) = \infty$ and $\{I_n:n=1,2,\ldots\}$, the number of immigrants into successive generations, are independent and identically distributed with probability generating function b satisfying $b(0) < 1$. Suppose also that $X_0 = 0$.

This paper describes the behavior of X_n as $n \to \infty$ in terms of appropriate limit theorems. In §2 we show that for a certain increasing function U constructed as in [6], the sequence $\{e^{-n}U(X_n)\}$ converges almost surely and that the limit is non-defective if a certain condition on the immigration distribution obtains and is essentially infinite otherwise. It is pointed out that, in the former case, $\{X_n\}$ may be classified in terms of regularity and irregularity exactly as in the case of no immigration [6].

In §3 we consider the case where $Y_n = e^{-n}U(X_n) \to \infty$ a.s. Limit theorems are obtained for Y_n which involve a non-linear norming by a slowly varying (SV) function. Theorem 2 involves no extra assumptions but has a rather strange appearance: neater versions are given in Corollaries 1-3 under conditions expressed in terms of $G(x) = 1 - b(\exp[-1/V(e^x)])$, where V is the functional inverse of U. These results are analogues of the families of limit theorems for the case $m < \infty$ which were presented in Part I [3], and some examples are given to show that the conditions in Corollaries 1-3 are not vacuous. Finally, it is shown that there is no sequence (c_n) of norming constants such that Y_n/c_n has a limit in distribution which is neither defective nor degenerate at the origin.

To complete the survey in Part I, it suffices to mention that a version of Theorem 1 was first proved in [2] under conditions on f ensuring that U could be chosen as a suitable power of $\log^+$ and only convergence in law was established. The conditions of this result were substantially relaxed in [1] to allow U to be SV at infinity and strictly increasing. A restricted and more opaque version of Corollary 1 was proved in [2].

2. ALMOST SURE CONVERGENCE

Let f_n denote the n^{th} functional iterate of f and h_n the functional inverse of $-\log f_n(e^{-s})$. Let $p_n(s) = \prod_{m=1}^{n} b\left(\exp(-h_m(s))\right)$ and $p(s) = \lim_{n\to\infty} p_n(s)$ for $0 < s < -\log q$ where $q = f(q) < 1$. Since $h_n(s) \downarrow 0$ $(n\to\infty)$ it is clear that either $p(s) > 0$ or $\equiv 0$. In the former case it is clear that $\{[\exp(-h_n(s)X_n)]/p_n(s)\}$ is a martingale and the martingale convergence theorem shows that for each $0 < s < -\log q$, $\exp(-h_n(s)X_n) \xrightarrow{a.s.} W(s)$, say, and $E\,W(s) = p(s)$. The behavior of $\{X_n\}$ can be analyzed in precisely the same way in which that of the corresponding process without immigration was analyzed in [6]. In particular the classification of points in $(0,-\log q)$ as regular and irregular carries over to the present case and if $T = \sup\{s \mid 0 < s < -\log q,\ W(s) < 1\}$ it follows that T is non-defective and $P(T \geq s_r) = p(s_r)$, $P(T = s_r) = 0$ for every regular point s_r.

Finally, for the function U constructed in [6], the arguments used in this reference show that $e^{-n}U(X_n) \xrightarrow{a.s.} U(T^{-1})$. In the complementary case, when $p(s) \equiv 0$, the martingale convergence theorem shows that $\exp(-h_n(s)X_n) \xrightarrow{a.s.} 0$ for each $0 < s < -\log q$. To proceed further we need the following details about U. It is shown in [6] that by starting with a fixed $s_0 \epsilon (0, -\log q)$ it is possible to construct U so that it is continuous on $[0,\infty)$, identically zero on $[0, 1/(-\log q)]$, positive and strictly increasing to infinity on $(1/(-\log q), \infty)$ and satisfies the relations

$$U(1/h_n(s_0)) = e^n, \quad U(1/h_n(s))/U(1/h_n(s_0)) = U(1/s) \qquad (1)$$
$$(n=0,1,\ldots)$$

Choose a sequence $\{s_m : m=1,2,\ldots\}$ such that $-\log q > s_m \downarrow 0$ $(m\to\infty)$. Let Ω_m be a subset of the basic probability space such that $P(\Omega_m) = 1$ and $h_n(s_m)X_n(\omega)\to\infty$ $(n\to\infty; \omega\epsilon\Omega_m)$. Thus on $\Omega' = \bigcap_{m\geq 1}\Omega_m$, $h_n(s_m)X_n(\omega)\to\infty$ for each m and this implies that eventually $X_n(\omega) \geq 1/h_n(s_m)$ and hence, from (1), eventually

$$e^{-n}U(X_n(\omega)) \geq U(1/s_m) \qquad (\omega\epsilon\Omega').$$

Letting $n\to\infty$ and them $m\to\infty$, we obtain all except the final assertion of the following theorem.

<u>Theorem 1</u>. <u>If</u> $p(s) > 0$ <u>then</u> $e^{-n}U(X_n) \xrightarrow{a.s.} U(1/T)$ <u>where</u> T <u>is defined above</u>, <u>and if</u> $p(s) = 0$, $e^{-n}U(X_n) \xrightarrow{a.s.} \infty$. <u>Furtheremore</u>, $p(s) > 0$ <u>iff</u>

$$\sum_{j=1}^{\infty} b_j \log^+ U(j) < \infty \qquad (2)$$

<u>where</u> $\{b_j\}$ <u>denotes the immigration distribution</u>.

The last assertion follows from these observations. Since U is strictly increasing, the relations (1) can be solved to obtain an explicit expression

$$h_n(s) = 1/V(e^n U(s^{-1})) \qquad (3)$$

for h_n, and the proof of Lemma 2 in [1] needs, with one exception, only trivial changes to show that (2) holds iff

$$\prod_{n=1}^{\infty} b\left[\exp\left(-1/V(e^n U(s^{-1}))\right)\right] > 0$$

where V is the functional inverse of U, whence the assertion in this case. The exception is that the proof given in [1] assumes that $\log U$ is SV: that this is so follows from

<u>Lemma 1</u>. <u>If</u> L <u>is SV at infinity then so is</u> $L(U(\cdot))$.

<u>Proof</u>. By virtue of the uniform convergence theorem for SV functions [7] it suffices to show that for each $\lambda > 0$, $U(\lambda x)/U(x)$ is bounded away from zero and infinity for all sufficiently large x. It suffices to consider the case $1 \le \lambda < \infty$. The proof to be given parallels that of Lemma 2.3.3 in [6]. Choose $s_0 \varepsilon (0,-\log q)$ as in Construction 2.3.1 of [6], and set $D = [h(s_0), s_0]$. The functions $g_n(s) = h_{n+1}(s)/h_n(s)$ are *continuous* on D, and, from equations (2) and (3) of [6], it follows that

$$g_n(s) \ge g_{n+1}(s) \ ; \quad g_n(s) \to 0 \quad \text{as} \quad n\to\infty \ : \tag{4}$$

hence, by Dini's theorem, this convergence is uniform on D. Thus for all $n \ge n_0$, chosen large enough, and all $s \varepsilon D$, $\lambda/h_n(s) \le 1/h_{n+1}(s)$. For each sufficiently large x construct n and $s \varepsilon D$ so that $x^{-1} = h_n(s)$ (see Construction 2.3.1, [5]). It follows that

$$1 < U(\lambda x)/U(x) \le U(1/h_{n+1}(s))/U(1/h_n(s)) = e \ ,$$

and the lemma now follows.

3. <u>THE CASE $p(s) \equiv 0$</u>

The description of the behavior of X_n in the case $p(s) \equiv 0$ is through distributional, rather than almost sure, limit theorems.

The first step in deriving them is to note that, from (3),

$$E\left(\exp(-h_n(t)X_n)\right) = p_n(t) = \prod_{m=1}^{n} b\left[\exp\left(-1/V(e^m U(t^{-1}))\right)\right] .$$

Using integral test comparisons, as in [3], it is not difficult to show that

$$p_n(t) = \Delta_n(t) \exp \int_0^n \log b\left[\exp\left(-1/V(e^y U(t^{-1}))\right)\right] dy$$

where $\Delta_n(t_n) \to 1$ if $t_n \to 0+$ as $n\to\infty$. Denoting the integral by J_n and making the change of variable $e^x = e^y U(t^{-1})$, it follows that if $t = t_n \to 0+$ then

$$-J_n \sim j_n = \int_{\log U(1/t_n)}^{n+\log U(1/t_n)} G(x)dx \tag{5}$$

where $G(x) = 1 - b\left(\exp(-1/V(e^x))\right)$. We are now in a position to approach the main limit theorem, by making a suitable choice of t_n, and by using the following version of Reuter's *Lemma 1 in [1]*.

<u>Lemma 2. Let</u> U <u>be constructed according to Construction 2.3.1 in [6] and in addition suppose it is strictly increasing on</u> $(1/(-\log q),\infty)$. <u>Let</u> y_n <u>be positive, increasing and satisfy the properties</u>

(a) $y_n(u)/y_n(v)\to\infty$ $(n\to\infty)$ <u>if</u> $u > v$ <u>and</u> $0 < c < u,\ v < d \le \infty$; <u>and</u>

(b) $e^n y_n(u)\to\infty$ $(n\to\infty;\ c < u < d)$.

<u>If, for a sequence of non-negative random variables</u> $\{W_n\}$ <u>there is a continuous function</u> $a(\cdot)$ <u>such that</u>

$$E\left[\exp\left(-W_n/V(e^n y_n(u))\right)\right] \to a(u) \qquad (c < u < d;\ n\to\infty)$$

<u>then</u>

$$P(e^{-n}U(W_n) \le y_n(u)) \to a(u).$$

<u>Proof</u>. Let $A_n = \{e^{-n}U(W_n) \le y_n(u)\} = \{W_n \le V(e^n y_n(u))\}$. Choose $u\varepsilon(c,d)$ and $c < u_1 < u < u_2 < d$. If $Y_n^{(i)} = \exp\left[-W_n/V(e^n y_n(u_i))\right]$ $(i=1,2)$: then arguing as in [1] we obtain

$$EY_n^{(1)} - \exp(-\lambda_n^{(1)}) \le P(A_n) \le (EY_n^{(2)}) \exp \lambda_n^{(2)}$$

where $\lambda_n^{(i)} = V(e^n y_n(u))/V(e^n y_n(u_i))$ $(i=1,2)$. If we can show that $\lambda_n^{(1)} \to \infty$ and $\lambda_n^{(2)} \to 0$, the assertion follows upon then letting $u_i \to u$.

We shall prove only that $\lambda_n^{(1)} \to \infty$. It follows from (a) that for each $t > 1$

$$\lambda_n^{(1)} \ge V(te^n y_n(u_1))/V(e^n y_n(u_1))$$

for all sufficiently large n. Let $m(n)$ be the integer part of $n + \log y_n(u_1)$. By (b) $m(n) \to \infty$ $(n \to \infty)$ and $e^{m(n)} \le e^n y_n(u_1) \le e^{m(n)+1}$ and hence, taking $t = e^2$,

$$\lambda_n^{(1)} \ge V(\exp\{m(n)+2\})/V\left[\exp(m(n)+1)\right] = h_{m(n)}(1/V(e))/h_{m(n)+1}(1/V(e)),$$

where we have used (3). Hence, from (4), $\lim_{n\to\infty} \lambda_n^{(1)} = \infty$.

Now, for $x \ge 1$, let $\Lambda(x) = \exp \int_0^{\log x} G(y)dy$, which is SV at infinity and strictly increasing. Furthermore, the ratio

$$\Lambda(x)/\Lambda(xe^n) = \exp\left\{-\int_{\log x}^{\log x+n} G(y)dy\right\}$$

$$= \exp\left(-\int_1^{e^n} G(\log(zx))dz/z\right)$$

increases with x on $(1,\infty)$ from m_n say, to unity, and since $p(t) \equiv 0$ iff $\int_0^\infty G(x)dx = \infty$, it follows that $m_n \to 0$ $(n\to\infty)$; moreover, for each fixed $x \ge 1$, $\Lambda(x)/\Lambda(xe^n)$ decreases to zero as $n\to\infty$. Hence, for any $0 < u < 1$ and all n sufficiently large, we may define $y_n(u) > 0$ uniquely in such a way that

$$\Lambda(y_n(u)) = u\Lambda(e^n y_n(u)) \ ,$$

where it follows that y_n is increasing on $(0,1)$, that $y_n(u)\to\infty$ $(n\to\infty)$ and that if $0<v<u<1$,

$$\frac{y_n(u)}{y_n(v)} = \frac{\Lambda^{-1}\left[u\Lambda(e^n y_n(u))\right]}{\Lambda^{-1}\left[v\Lambda(e^n y_n(v))\right]} \geq \frac{\Lambda^{-1}\left[u\Lambda(e^n y_n(v))\right]}{\Lambda^{-1}\left[v\Lambda(e^n y_n(v))\right]} \to\infty$$

as $n\to\infty$, since Λ is SV at infinity. Choosing $U(t_n^{-1}) = y_n(u)$ in (5) we see that $j_n = u$ and hence $E\left[\exp\left(-X_n/V(e^n y_n(u))\right)\right] \to u \ (0<u<1)$, and that the other conditions of Lemma 2 are satisfied. This completes the proof of the following theorem.

<u>Thereom 2</u>. <u>If</u> $p(s) \equiv 0$, <u>then</u>

$$\Lambda(e^{-n}U(X_n))/\Lambda(U(X_n)) \xrightarrow{d} W$$

<u>where</u> W <u>is uniformly distributed on</u> $[0,1]$.

Analogues of Theorem 2 also obtain when $m<1$ and $1<m<\infty$; see [3].

The proof of Theorem 2 has been carried out under the condition $X_0 = 0$. However

$$\begin{aligned} P_i^{(n)}(t) &= E\left[\exp(-X_n h_n(t)) \mid X_0=i\right] \\ &= \left[f_n\left(\exp(-h_n(t))\right)\right]^i P_0^{(n)}(t) \\ &= e^{-it}P_0^{(n)}(t), \end{aligned}$$

and since t was chosen to converge to zero, we see that Theorem 2 is valid for any initial state. Furthermore, $\Lambda(e^{-n}U(\cdot))/\Lambda(U(\cdot))$ is strictly increasing and continuous and hence the weakly convergent sequence in Theorem 2 is a Markov chain and, in addition, is mixing [5, Theorem 2]. Thus [4] we cannot have convergence in probability in Theorem 2.

The quantity converging in law in Theorem 2 has a rather odd appearance since $U(X_n)$ is present in both numerator and denominator. Tidier versions can be obtained by making estimates of

$$\Lambda\left(e^{-n}U(X_n)\right)/\Lambda\left(U(X_n)\right) = \exp\left\{-Y_n \int_{1-n/Y_n}^{1} G(zY_n)\,dz\right\} ; \qquad (6)$$

under appropriate assumptions on the behavior of G. For brevity, we write $Y_n \equiv \log U(X_n)$ throughout.

<u>Corollary 1</u>. If $p(s) \equiv 0$ and $\lim_{x\to\infty} xG(x) = 0$, then

$$\Lambda\left(e^{-n}U(X_n)\right)/\Lambda(e^n) \xrightarrow{d} W ,$$

where W is uniformly distributed on $[0,1]$.

<u>Proof</u>. Since $xG(x) \to 0$, we have

$$\int_{Y_n-n}^{Y_n} G(y)\,dy = o\left\{-\log(1-n/Y_n)\right\} ; \qquad (7)$$

combining Theorem 2, (6) and (7), it follows that, as $n\to\infty$, $n/Y_n \xrightarrow{d} 1$. Hence, as $n\to\infty$,

$$\Lambda\left(U(X_n)\right)/\Lambda(e^n) = \exp\left\{\int_n^{Y_n} G(y)\,dy\right\}$$

$$= \exp\left\{o(\log[Y_n/n])\right\} \xrightarrow{d} 1 ,$$

and the result follows.

<u>Corollary 2</u>. If $p(s) \equiv 0$ and $\lim_{x\to\infty} xG(x) = a$, $0 < a < \infty$, then

$$n^{-1}\log U(X_n) - 1 \xrightarrow{d} W ,$$

where $P[W \leq u] = \{u/(1+u)\}^a$.

Proof. Immediately, from (6), we have

$$\Lambda(e^{-n}U(X_n))/\Lambda(U(X_n)) \overset{\text{a.s.}}{\sim} \{1 - n/Y_n\}^a ,$$

and the result now follows from Theorem 2.

Corollary 3. If $p(s) \equiv 0$, $\lim_{x\to\infty} xG(x) = \infty$, and G is regularly varying at infinity, then $nG(\log U(X_n)) \xrightarrow{d} W$, where W is a standard negative exponential random variable.

Proof. Here, from (6), we have

$$\Lambda(e^{-n}U(X_n))/\Lambda(U(X_n)) = \exp\left\{ -Y_n \int_{1-n/Y_n}^{1} G(zY_n)\, dz \right\} \; ;(8)$$

combining (8) with Theorem 2 and $\lim_{x\to\infty} xG(x) = \infty$, it follows that, as $n\to\infty$, $n/Y_n \xrightarrow{d} 0$. Hence, since G varies regularly,

$$\Lambda(e^{-n}U(X_n))/\Lambda(U(X_n)) \sim \exp\{-nG(Y_n)\} ,$$

and the result follows.

If in Corollary 3 G has a positive index Δ, then $0 < \Delta \leq 1$ and the conclusion can be transformed to the form

$$a_n^{-1} \log U(X_n) \xrightarrow{d} W'$$

where W' has the extreme value distribution function $\exp(-x^{-\Delta})$ and $G(a_n) = n^{-1}$.

We now show that for any U the hypotheses of Corollaries 1-3 can be satisfied. Let A be a positive integer valued random variable, define $I = [V(A)]$ and let b be the probability generating function of I. Since

$$V(A) - 1 \leq I \leq V(A) \qquad (9)$$

and $\log U$ is SV at infinity, it follows that condition (2) is satisfied iff $E\log^+ A < \infty$. In [3,§3.1] several examples were given where $E\log^+ A = \infty$, and for each of these $T(x) = P(A > x)$ is SV at infinity. We now suppose this to be always the case. It follows then, from Lemma 1 and (9), that $P(I > x) \sim T(U(x))$ $(x \to \infty)$ and hence an Abelian theorem for power series yields

$$(1-b(s))/(1-s) = \sum s^j P(I > j) \sim (1-s)^{-1} T(U(1-s)) \quad (s \uparrow 1).$$

Letting $s = \exp(-1/V(e^x))$ and invoking Lemma 1 once again, we obtain

$$G(x) \sim T(e^x). \tag{10}$$

Let $\log_1 x = \log x$ and $\log_k x = \log(\log_{k-1} x)$ $(k=2,3,\ldots)$ for all sufficiently large x. In [3] an example was given for which

$$T(x) \sim c[\prod_{k=1}^{r} \log_k x]^{-1} \qquad (x \to \infty)$$

where $r \geq 2$ and c is a certain constant. Using (10) it is obvious that $\int_0^\infty G(x)dx = \infty$ and that $xG(x) \to 0$ $(x \to \infty)$ and hence this example satisfies the conditions of Corollary 1.

Discrete distributions were also constructed in [3] for which

$$T(x) \sim a/\log x \qquad (0 < a < \infty);$$

$$T(x) \sim c(\log x)^{-(\delta-1)} \qquad (0 < c < \infty,\ 1 < \delta < 2);$$

and

$$T(x) \sim (c/b)(\log_r x)^{-b} \qquad (0 < b, c < \infty,\ r \geq 2).$$

Using (10) we see that these examples satisfy the hypotheses of Corollaries 2 and 3.

Finally, we show that the neatest version of Theorem 2 that could be hoped for is in fact impossible.

Theorem 3. There is no sequence of constants (c_n) such that $c_n^{-1} U(X_n)$ has a limit in distribution which is neither defective nor degenerate at zero.

Proof. For any $0 < x_1 < x_2 < \infty$, let $p_n(x_1,x_2) \equiv P[x_1 \le c_n^{-1}U(X_n) \le x_2]$. Then, since $\Lambda(e^{-n}U(\cdot))/\Lambda(U(\cdot))$ is continuous and strictly increasing, it follows also that

$$p_n(x_1,x_2) = P[r_n(x_1) \le \Lambda(e^{-n}U(X_n))/\Lambda(U(X_n)) \le r_n(x_2)] ,$$

where

$$0 < r_n(x) \equiv \Lambda(e^{-n}c_n x)/\Lambda(c_n x) < 1.$$

If (c_n) is such that $c_n^{-1}U(X_n)$ is to converge in distribution, we must have $c_n e^{-n} \to \infty$ because of Theorem 1. Hence, since Λ is SV, we see that $r_n(x_2) = r_n(x_1)[1+o(1)]$ as $n\to\infty$. Choose any subsequence (n_k) such that $r_{n_k}(x_1) \to r$ for some $r\epsilon[0,1]$. Then, for any $\epsilon > 0$, the intervals $\left[r_{n_k}(x_1), r_{n_k}(x_2)\right]$ belong to $(r-\epsilon, r+\epsilon)$ for all k sufficiently large. It follows from Theorem 2 that $p_{n_k}(x_1,x_2) \to 0$, and hence that, if $c_n^{-1}U(X_n)$ converges in distribution, its limit puts no mass on $(0,)$.

It is interesting to note the contrast between the cases $p(x) > 0$ and $p(x) \equiv 0$. In the former, the asymptotic behavior is dominated by the underlying Galton-Watson process, and the effect of immigration, apart from preventing extinction, is seen only in the distribution of the limit of $U(X_n)e^{-n}$: eventually, the contribution of the immigration process becomes negligible. However, when $p(x) \equiv 0$, the immigration distribution has such a broad tail that $U(X_n)e^{-n}$ is pushed off to infinity a.s. by the infinite sequence of occasional, but very large, inflows of immigrants. The character of Theorem 2 and its Corollaries, giving limits in distribution but not with probability one, reflects the nature of the immigration process rather than that of the Galton-Watson process. In particular, unlike the case when $p(x) > 0$, the limiting distribution appearing in Theorem 2 is the same, whether or not the Galton-Watson process is regular or irregular.

* * * * *

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sub n

(e to the -nth power)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number) This paper obtains some limit theorems for the simple branching process allowing immigration $\langle X_n \rangle$ when the offspring mean is infinite. It is shown that there exists a function U such that $\langle e^{-n} U(X_n) \rangle$ converges almost surely and if $s = \Sigma b_j \log^+ U(j) < \infty$, where $\{b_j\}$ is the immigration distribution, the limit is non-defective and non-degenerate but is infinite if $s = \infty$. When $s = \infty$ limit theorems are found for $\langle U(X_n) \rangle$ which involve a slowly varying non-linear norming.

Sum over b sub j log(+)

sub n

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sub j

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